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A Comparative Study of Nonlinear Growth Models on Teak (*Tectonagrandis L.*) in India

Research article

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Abstract

This study presents a comparative study of the most commonly used six growth models Monomolecular, Gompertz, Logistic, Weibull, Von Bertalanffy and Chapmen Richard growth models for describing the growth pattern of Teak (*Tectonagrandis L.*) in India. Various methods of estimation are introduced to estimate the parameters of above mentioned model. Four sets of well-established teak data of India have been used for testing the validity of the models. The best fit model has been selected based on a selection criterion. According to the results of our calculation, it can be concluded that, the three parameter monomolecular growth model is more reasonable over the remaining growth models to describe the growth of Teak in India.

Keywords: Growth model; Chapmen richards; Von bertalanffy; Logistic; Gompertz; Monomolecular; Weibull; Teak growth; Method of estimation

Introduction

Teak (*Tectonagrandis L.*) is an all-around premier species of many favorable properties and will remain as one of the most admired and precious tree. Teak plant has a very economical importance as its wood is very durable, resistant to fungi. It is indigenous to only four countries namely India, Myanmar, Thailand and Lao People's Democratic Republic [1]. In this study an attempt has been made to analyze the growth (height and Diameter at breast height) of Teak in India with the help of a set of suitable growth functions.

The Chapmen Richards growth model along with its limiting cases has a wide application in forestry. Here we consider five limiting cases namely Von Bertalanffy, Monomolecular, Logistic and Gompertz growth model. Various researchers have been used these models and Weibull growth model in forestry. This study presents a comparative study of these models to describe the growth pattern of Teak plant in India.

This paper is organized as follows, Section II gives an overview of the data and methodology for the estimations of the parameters of the model. In this paper we apply some new technique to estimate the parameters of Von Bertalanffy and Chapmen Richard growth models. The main advantages of the new techniques applied in this paper are less computation and can be used for any growth data. The parameters of Monomolecular, Logistic and Gompertz growth models are estimated using the technique used by Borah and Mahanta [2]. The generalized Newton Raphson technique used in Mahanta and Borah are applied to estimate the parameters of Weibull growth model [3].

In section III, we describe the selection criteria for selecting the best fit model. The selection criteria consist of six distinct steps.

The two final sections (section IV and V) include a brief analysis of the results and some of the main conclusions.

Materials and Methods

The integral forms of Monomolecular, Gompertz, Logistic, Weibull (two, three and four parameters), Von Bertalanffy (two, three and four parameters) and Chapmen Richard (three and four parameter) growth models are shown in Table 1. The data composed in the Table 2 are of height and DBH growth data from Teak trees in Warangal state and Hoshangabad division of India [1].

Here A,B,K,d,β,b_1,b and m are parameters to be estimated, y is the dependent growth variable, t is the independent variable and exp(e) is the base of the natural logarithms. The parameters of the growth models are defined as: A is the asymptote; K is the parameter governing the rate at which the regress and approaches its potential maximum; m is the allometric constant; d is the instant rate of growth in the inflection point, b is the value of the growth at the initial age and B, β and b_1 are biological constants.

The different growth models can be written in the form as

$$y_i = f(t_i, B) + \varepsilon_i, \tag{1}$$

Table 1: The integral forms of the growth models along with the source.

S/N	Common growth models	Integral form of the models (y(t))	Source
1	Monomolecular	$y_i = f(t_i, B) + \varepsilon_i,$	[2]
2	Gompertz	$Ae^{-Be^{-Kt}}$	
3	Logistic	$\frac{A}{1+\beta e^{-Kt}}$	
4	Weibull 4 parameter	$A - B\,exp(-Kt^m)$	[3]
5	Weibull 3 parameter	$A(1-\exp(-Kt^m))$	
6	Weibull 2 parameter	$A(1-exp(-Kt))$	
7	Chapman Richards 4 parameter	$A\{1-Be^{-Kt}\}^d$	[5]
8	Chapman Richards 3 parameter	$A\{1-e^{-Kt}\}^d$	[7]
9	von Bertalanffy 4 parameters	$\{A^{1-m} - b_1 e^{-Kt}\}^{\frac{1}{1-m}}$	[8]
10	von Bertalanffy 3 parameters	$A-(A-b)e^{-Kt}$	[9]
11	von Bertalanffy 2 parameters	$A(1-e^{-Kt}).$	[10]

Table 2: Height and DBH growth data from Teak trees in Warangal state and Hoshangabad division of India.

Age (Years)	Warangal state		Hoshangabad division	
	eight(m)	BH(cm)	eight(m)	BH(cm)
10	0.3	2	0.7	0.8
20	2.6	2	0.9	3.2
30	6.7	0	0.7	0.8
40	0	6	2.5	7.4
50	2.4	9	4	2.5
60	4.3	8	5.5	6.3
70	--	1	7.1	9.1
80	--	--	--	1.7
90	--	--	--	3.9

i=1,2,⋯,n, where B is the vector of parameters b_j (b_1,b_2⋯,b_s) to be estimated, s is the number of parameter, n is the number of observations and ε_i's are random errors in the models has mean zero and constant variance σ^2. The following methods of estimation have been used to fit the growth models.

Method of Estimation

The parameters of Monomolecular, Logistic and Gompertz growth have been estimated using the technique used by Borah and Mahanta [2]. The generalized Newton Raphson technique used in Mahanta and Borah are applied to estimate the parameters of Weibull growth models [3]. The new methods for the estimation of the parameters of Von Bertalanffy and Chapmen Richard growth models are described as follows.

Methods to estimate the parameters of Chapmen Richards's growth model

Fitting of the four parameters model

Method I: In this method, first assume that the parameter d is known from its definition. Assume n be the total number of observation and let t_a,t_b and t_c are any three observations from the set of data. Then for i = a,b,c the Chapmen Richards growth model with four parameters can be written as

$$\ln y_i = \ln A + d\ln\left(1 - Be^{-Kt_i}\right) \tag{2}$$

Now,

$$\ln y_a - \ln y_b = d\ln\left\{\frac{1-Be^{-Kt_a}}{1-Be^{-Kt_b}}\right\} \tag{3}$$

and

$$\ln y_b - \ln y_c = d\ln\left\{\frac{1-Be^{-Kt_b}}{1-Be^{-Kt_c}}\right\} \tag{4}$$

From the equation (3) and (4),

$$\left(A_1A_2 - A_2\right)x^{t_c} + \left(1 - A_1A_2\right)x^{t_b} + \left(A_2 - 1\right)x^{t_a} = 0 \tag{5}$$

Where $A_1 = \exp\frac{\ln y_a - \ln y_b}{d}$ and $A_2 = \exp\frac{\ln y_b - \ln y_c}{d}$

The equation (5) can be solved using any iteration method, and then the parameter K can be estimated as, $\hat{K} = \ln\frac{1}{x}$.

After estimation of the parameter K; the parameters B, A and d can be estimated using the equations (2), (3) and (4). The required estimated parameters are given by

$$\hat{B} = \frac{1 - \exp\frac{\ln y_b - \ln y_c}{d}}{\exp(-Kt_b) - \exp\frac{\ln y_b - \ln y_c}{d}\exp(-Kt_c)},$$

$$\hat{A} = \exp\left\{\ln y_c - r\ln\left(1 - Be^{-Kt_c}\right)\right\},$$

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$$\hat{d} = \frac{\ln y_a - \ln A}{\ln\left(1 - Be^{-Kt_a}\right)}$$

If the data set are equidistant then we may take $r = \left[\frac{n}{3}\right], t_a = r, t_b = 2r$ and $t_c = 3r$ Then the equation obtained by solving equations (3) and (4) will be in quadratic form with $x = e^{-rK}$.

Method II: In this method, assume that the parameters B and K are known. Let $S_1 = \Sigma_{i=1}^{r} ln\, y_i$ and $S_2 = \Sigma_{i=r+1}^{2r} ln\, y_i$, where n be the total number of observation and $r = \left[\frac{n}{2}\right]$, then the estimated parameters are given by

$$\hat{d} = \frac{S_2 - S_1}{\left\{\ln\left(\prod_{i=r+1}^{2r}\left(1 - Be^{-Kt_i}\right)\right) - \ln\left(\prod_{i=1}^{r}\left(1 - Be^{-Kt_i}\right)\right)\right\}},$$

$$\hat{A} = \exp\left\{\frac{S_1 - d\ln\left(\prod_{i=1}^{r}\left(1 - Be^{-Kt_i}\right)\right)}{r}\right\}$$

After estimating the parameters A and d the parameter K and B can be estimated as

$$\hat{K} = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln\left[\frac{\prod_{i=1}^{r}\left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)}{\prod_{i=r+1}^{2r}\left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)}\right],$$

$$\hat{B} = \exp\left\{\frac{K}{n}\sum_{i=1}^{n} t_i + \frac{1}{n}\ln\left\{\prod_{i=1}^{n}\left(1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right)\right\}\right\}.$$

Fitting of the three parameters model

Method I: In this method, let n be the total number of observation and let $r = \left[\frac{n}{3}\right], t_a = r, t_b = 2r$ and $t_c = 3r$. Then for i = a,b,c; the Chapmen Richards growth model with three parameters can be written as

$$üüy_i = \quad A + d \quad \left(-e^{-Kt_i}\right) \tag{6}$$

Now

$$\ln y_a - \ln y_b = d \ln\left\{\frac{1 - e^{-Kt_a}}{1 - e^{-Kt_b}}\right\} \tag{7}$$

Similarly

$$\ln y_c - \ln y_b = d \ln\left\{\frac{1 - e^{-Kt_c}}{1 - e^{-Kt_b}}\right\} \tag{8}$$

By solving the equations (6), (7) and (8); the estimated parameters are given by

$$\hat{K} = \frac{\ln y_c - \ln y_b}{r(\ln y_b - \ln y_a)}, \quad \hat{d} = \frac{\ln y_a - \ln y_b}{\ln\frac{1 - e^{-Kt_a}}{1 - e^{-Kt_b}}}$$

$$\hat{A} = \exp\left\{\ln y_c - r\ln\left(1 - e^{-Kt_c}\right)\right\}.$$

Where y_a, y_b and y_c are respective observations at time t_a, t_b and t_c respectively.

Method II: For this method, first assume that the parameter K is known from the method I. Let $S_1 = \Sigma_{i=1}^{r} ln\, y_i$ and, $S_2 = \Sigma_{i=r+1}^{2r} ln\, y_i$, where n be the total number of observation and let $r = \left[\frac{n}{2}\right]$. Then the Chapmen Richards growth model with three parameters can be written as

$$\ln y = \ln A + d\ln(1 - e^{-Kt}) \tag{9}$$

Now,

$$S_1 = r\ln A + d\ln\left\{\prod_{i=1}^{r}(1 - e^{-Ki})\right\} \tag{10}$$

and

$$S_2 = r\ln A + d\ln\left\{\prod_{i=r+1}^{2r}(1 - e^{-Ki})\right\} \tag{11}$$

Now by solving the equations (9), (10) and (11); the estimated parameters are given by

$S_2 = \sum$

$$\hat{d} = \frac{S_2 - S_1}{\left\{\ln\left(\prod_{i=r+1}^{2r}\left(1 - e^{-Kt_i}\right)\right) - \ln\left(\prod_{i=1}^{r}\left(1 - e^{-Kt_i}\right)\right)\right\}},$$

$$\hat{A} = \exp\left\{\frac{S_1 - d\ln\left(\prod_{i=1}^{r}\left(1 - e^{-Kt_i}\right)\right)}{r}\right\}$$

After estimating the parameters A and d, the parameter K can be estimated as

$$\hat{A} = \exp\left\{\frac{S_1 - d\ln\left(\prod_{i=1}^{r}\left(1 - e^{-Kt_i}\right)\right)}{r}\right\}.$$

After estimating the parameters *A* and *d*, the parameter *K* can be estimated as

$$\hat{K} = \frac{1}{n}\ln\left[\prod_{i=1}^{n}\frac{1}{\left\{1 - \left(\frac{y_i}{A}\right)^{\frac{1}{d}}\right\}^{\frac{1}{t_i}}}\right]$$

Methods to estimate the parameters of Von Bertalanffy growth model

Estimation of four parameters von Bertalanffy growth model

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The von Bertalanffy growth model can also be written as

$$y = \left(A^{1-m} - b_1 e^{-Kt}\right)^{\frac{1}{1-m}},$$

$$y = A\left(1 - Be^{-kt}\right)^{d}. \quad (12)$$

Where $B = \frac{b_1}{A^{1-m}}$ and $d = \frac{1}{1-m}$

The equation (12) is in the form of Chapmen Richards's growth model and its parameters can be estimated using the same methodology as Chapmen Richards's growth model. After estimating the parameters B and d the required parameters can be estimated using the relations $b_1 = BA^{1-m}$ and $m = \frac{d-1}{d}$.

Estimation of three parameters von Bertalanffy growth model

The Von Bertalanffy growth model with three parameters can also be written as

$$y(t) = A\left(1 - b_2 e^{-Kt}\right), \quad (13)$$

Where $b_2 = \frac{A-b}{A}$

Method I: Let n be the total number of observation and let t_a, t_b and t_c are any three observations from the set of data. Then for i = a,b,c; the equation (13) can be written as

$$\ln y_i = \ln A + \ln\left(1 - b_2 e^{-Kt_i}\right). \quad (14)$$

Now,

$$\ln y_a - \ln y_b = \ln\left\{\frac{1 - b_2 e^{-Kt_a}}{1 - b_2 e^{-Kt_b}}\right\}, \quad (15)$$

and

$$\ln y_b - \ln y_c = \ln\left\{\frac{1 - b_2 e^{-Kt_b}}{1 - b_2 e^{-Kt_c}}\right\}. \quad (16)$$

From the equation (3) and (4), one can obtain an equation of the form

$$\left(A_1 A_2 - A_2\right)x^{t_c} + \left(1 - A_1 A_2\right)x^{t_b} + \left(A_2 - 1\right)x^{t_a} = 0, \quad (17)$$

Where $A_1 = \exp(\ln y_a - \ln y_b)$ and $A_2 = \exp(\ln y_b - \ln y_c)$

The equation (5) can be solve using any iteration method, then the parameter K can be estimated as, $K = \ln\frac{1}{x}$.

After estimating the parameter K; the parameters b_2 can be estimated from the equation (4) and then using the relation $b = A - Ab_2$, one can estimate the parameter b. To estimate the parameter A the following relation can be used

$$A = \exp\left\{\ln y_c - \ln\left(1 - b_1 e^{-Kt_c}\right)\right\}.$$

For some equidistant data set, one may consider $r = \left[\frac{n}{3}\right], t_a = r, t_b = 2r$ and $t_c = 3r$. In this case the equation obtained by solving equations (3) and (4) will be in quadratic form with $x = e^{-rK}$.

Method II: Let n be the number of observations and $r = \left[\frac{n}{2}\right]$. Assume that the parameter A is known. Then for i=1,2,⋯,n; the model form (13) can be written as:

$$\ln b_2 = Kt_i + \ln\left\{1 - \frac{y_i}{A}\right\}. \quad (18)$$

For the first and second d observations, the sum can be expressed as;

$$r \ln b_2 = K\,\Sigma_{i=1}^{r} t_i + \ln\left\{\Pi_{i=1}^{r}\left\{1 - \frac{y_i}{A}\right\}\right\}, \quad (19)$$

$$r \ln b_2 = K\Sigma_{i=+1}^{2r} t_i + \ln\left\{\Pi_{i=r+1}^{2r}\left\{1 - \frac{y_i}{A}\right\}\right\}. \quad (20)$$

From the equation (19) and (20); the parameters K can be estimated and which is

$$K = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln \frac{\prod_{i=1}^{r}\left\{1 - \frac{y_i}{A}\right\}}{\prod_{i=r+1}^{2r}\left\{1 - \frac{y_i}{A}\right\}}. \quad (21)$$

After estimating K, the parameter b_2 can be estimated by considering sum of (18) for all i=1,2,⋯,n; that is

$$b_2 = \exp\left[\frac{K}{n}\Sigma_{i=1} t_i + \frac{1}{n}\ln\left\{\Pi_{i-1}^{n}\left\{1 - \frac{y_i}{A}\right\}\right\}\right]. \quad (22)$$

Then the parameter b can be estimated using its definition with the parameter b_2. Again to estimate the parameter A, one can use the estimated value of K and b_2;

$$A = \left\{\Pi_{i=1}^{n} \frac{y_i}{1 - b_2 e^{-Kt_i}}\right\}^{\frac{1}{n}}. \quad (23)$$

Estimation of two parameters von Bertalanffy growth model

Method I: Let the total number of observation is n. Let t_1 be the first observation and t_2 be the n^{th} observation. Then the two parameters von Bertalanffy growth model can be written as

$$y_1 = A\left(1 - e^{-K}\right), \quad (24)$$

$$y_2 = A\left(1 - e^{-Kn}\right), \quad (25)$$

Now by solving equation (24) and (25), we have

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$$y_1 x^n - y_2 x + \left(y_2 - y_1\right) = 0. \tag{26}$$

Where $x = e^{-K}$. The equation (26) can be solve using any iteration method. After finding the value of x, the parameter can be estimated using $K = -\ln x$. Then the parameter A can be estimated using the equation (24) or (25). Now since the iteration method need an initial value. To get the initial value one can use the following procedure:

For the first and second data of the data set, the equation (26) can be written as

$$y_1 x^2 - y_2 x + \left(y_2 - y_1\right) = 0 \tag{27}$$

Which is a quadratic equation and by solving it one have two values of x. The non-negative value (s) of x can be used as starting value.

Method II: In this assume that the parameter A is known from the previous method. Then rewriting the model form in terms of K and then considering the sum of all observations, we have

$$K = \frac{1}{n}\ln\left[\Pi_{i-1}^{n} \frac{1}{\left(1-\frac{y_i}{A}\right)^{\frac{1}{t_i}}}\right]; i = 1,2,\cdots,n \tag{28}$$

After estimating K, again rewriting the model in terms of A and adding for the entire observations, the parameter A can be estimated, that is

$$A = \frac{1}{n}\Sigma_{i=1}^{n}\frac{y_i}{1-e^{-Kt_i}};\ i = 1,2,\cdots,n \tag{29}$$

Selection Criteria of Best Fit Model

After fitting the growth models using different methods of estimation, we select the best fit model based on the following selection criteria. The selection criteria consist of six distinct steps.

Step I: Logical and biological consistency: In this step, we checked the logical consistent and biologically realistic of the estimated parameters. The growth models with non-consistent and non-natural consistency and poor statistical properties are excluded.

Step II: Chi-Square Goodness-of-Fit Test (χ^2): This test enables us to see how well does the growth model fit to the observed data. The Chi-Square is defined as

$$\chi^2 = \sum_{i=1}^{n}\frac{\left(y_i - \hat{y}_i\right)^2}{\hat{y}_i},$$

Where y_i is observed value and $\hat{y}_i$ is the predicted value for i=1,2,⋯,n. If the calculated value of χ^2 is greater than the tabulated value of χ^2 with n-1-p degrees of freedom (where p is the number of parameters of the growth model and n is the number of observations) then the null hypothesis is rejected otherwise accepted. In this literature, only those results will considered which have 95% level of significance with their respective degree of freedom.

Step III: The Root Mean Square Error (RMSE): The RMSE measures to aggregate the residuals into one measure of predictive power. The RMSE of a model prediction with reference to the calculable variable is defined as:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{n}},$$

Where y_i is observed values and $\hat{y}_i$ is the predicted values for i=1,2,⋯,n. By comparing the RMSE we select the ten best results.

Step IV: Coefficient of determination (R^2) and adjusted coefficient of determination (R_a^2): Next we evaluate the coefficient of determination (R^2). The R^2 value indicates how well data point fits a growth model. Generally the value of (R^2) lies between 0 and 1($0 \le R^2 \le 1$). But it is impossible for R^2 to actually attain 1, if pure error exists. In practice, sometime negative value of R^2 may occur. Theoretically the value 1 indicates a perfect fit, 0 reveals that the model is not a better than the simple average and negative value indicate a poor model [4]. If the value of R^2 is above 0.9, it is accepted as efficient [5]. The mathematical formulation of the coefficient of determination is,

$$R^2 = 1 - \frac{\sum\left(y_i - \hat{y}_i\right)^2}{\sum\left(y_i - \bar{y}\right)^2},$$

Where $\bar{y}$ is the mean of the response variables.

The R_a^2 value is an endeavor to redress the propensity for over fitting of R^2 by adjusting both the numerator and the exterminator by their respective degree of freedom and define as

$$R_a^2 = 1 - \left(1 - R^2\right)\left(\frac{n-1}{n-p}\right).$$

The R_a^2 can be used to compare growth models not only to a specific set of data but also to two or more entirely different sets of data. The equation with the least standard error of the estimate will most likely also have the maximum R_a^2. In this manuscript, only those results will consider which have R_a^2 value not less than 0.99.

Step V: Confidence interval: In this step, we find the confidence intervals of the estimated parameters. Let Bis the vector of the parameters (say the parameters are $\beta_1, \beta_2, \cdots, \beta_p$) of the growth models. Confidence limits for the true value of the parameters B can be evaluated on the basis of the linearized approximation, evaluated at the predicted value of the parameters $\hat{B}$. The $100(1-\alpha)\%$ confidence interval for the parameters B is

$$\hat{B}_i \pm t_{\frac{\alpha}{2},n-p}\, se\left(\hat{B}_i\right),$$

Where $t_{\frac{\alpha}{2},n-p}$ is the t-value at n-p degrees of freedom and

$$se\left(\hat{B}_i\right)=\left\{appropriate\,diagonal\,element\,of\left(\hat{Z}'\hat{Z}\right)^{-1}S^2\right\}^{\frac{1}{2}}$$ and

Here $S^2 = S(\hat{B})/(n-p); S\left(\hat{B}\right)=\Sigma_{i=1}^{n}\left\{y_i - f(t_i,\hat{B})\right\}^2$

$$\hat{Z}=\begin{bmatrix} \frac{\partial f(t_1,B)}{\partial\beta_1} & \frac{\partial f(t_1,B)}{\partial\beta_2} & \cdots & \frac{\partial f(t_1,B)}{\partial\beta_p} \\ \frac{\partial f(t_2,B)}{\partial\beta_1} & \frac{\partial f(t_2,B)}{\partial\beta_2} & \cdots & \frac{\partial f(t_2,B)}{\partial\beta_p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f(t_n,B)}{\partial\beta_1} & \frac{\partial f(t_n,B)}{\partial\beta_2} & \cdots & \frac{\partial f(t_n,B)}{\partial\beta_p} \end{bmatrix}.$$

The final estimate of the parameters with ~95% confidence band excluding zero, indicating that there are only non-zero values of the parameters and then they are always significant. In this step, those results with negative confidence interval have been eliminated.

Step VI: Approximate R^2 for prediction: Finally we calculate the approximate R^2 for prediction and it is given by

$$R^2_{prediction}=1-\frac{PRESS}{\sum\left(y_i-\bar{y}\right)^2},$$

Where $PRESS=\Sigma_{i=1}^{n}\left(\frac{e_i}{1-h_{ii}}\right)^2$ is known as the PRESS statistic.

Here $e_i = y_i - \hat{y}_i$ and h_{ii} are the diagonal elements of hat matrix $H=T(T'T)^{-1}T'$ and T is a $n\times 1$ matrix of the independent variables. This statistic gives some indication of the predictive capability of the model.

If the value of $R^2_{prediction}$ is r and the value of R^2 is m, then we could expect from the model to explain about r% of the variability in predicting new observations, as compared to the approximately m% of the variability in the original data explained by the fitting [6]. Base on this statistics we try to select the best fit model for different growth of teak in India.

Results and Discussion

The eleven different forms of six growth models have been fitted to height and DBH growth data from Teak trees in Warangal state and Hoshangabad division of India. The parameters of these models are estimated using a total of thirty one methods of estimation.

The estimation of parameters for the growth models along with the summary of statistical analysis to height growth data from Hoshangabad division are presented in Table 3. For the height growth data from Hoshangabad division, Weibull model with four parameters unable to provide a fit due to having a singular matrix in the denominator during computation. Based on six model selection criteria as discussed above we summarized the results as bellow [7-10].

Step I: The Gompertz growth model fitted by method II and IV, Logistic model estimated by method II, IV, V and VI are rejected due to non-logical estimation of the parameters. All the methods have estimated the asymptotes smaller than the dominant height of Teak tree (17.10 m). The estimated parameters of the rest of the models are logically consistent and biologically significant.

Step II: Based on step II, Gompertz growth model (method V), Logistic model (method I and III), Von Bertalanffy four parameters model (methods I and II), Chapmen Richards four parameters model (methods I and II) and Chapmen Richards three parameters model (method I) are rejected due to having less than 95% level of significance.

Step III: Considering the relative value of RMSE, the ten best results have been selected in this step. Comparing the value of RMSE, Monomolecular growth models with all its methods of estimation, Gompertz growth model with method VI, Weibull two and three parameters growth models along with Chapmen Richards three parameters model for method II are promoted for the next level.

Step IV: In the fourth step, Monomolecular growth model with method II, Gompertz growth model with method VI, Weibull two parameters growth model are eliminated as they have R_a^2 value less than 0.99.

Step V: All surviving results along with the 95% confidence level are demonstrated in Table 4. It is observed that all parameters for all candidate growth models are significantly different from zero.

Step VI: The sixth and final selection criteria is based on R^2 and $R^2_{prediction}$, as this statistic gives some indication of the predictive capability of the growth models. From the final step, we select the best growth model. In case of height growth data from Hoshangabad division, the monomolecular growth model (methods VI) is found to be more suitable as the value of $R^2_{prediction}$ and R^2 (99.34 and 99.58 respectively) are better than the remaining surviving growth models. The observed and the estimated value are shown in Figure 1. The eliminated results in each step are highlighted accordingly in the Table 3.

The estimation of parameters for the growth models and the summary of statistical analysis to DBH growth data from Hoshangabad division are presented in Table 5. In this case, three parameters Von Bertalanffy growth model (method I and II) and two parameters Von Bertalanffy growth model (method I and II) are rejected due to non-logical estimation of the parameters. In all the cases, some of their parameters estimate negative value, which violate the model assumption concerning the parameters. Logistic model (method II,V and VI) is also eliminated due to having the estimates of asymptotic parameters smaller than the dominant DBH of Teak tree (43.90 cm). The eliminated results in each step are also highlighted accordingly in the Table 6. In the case of DBH growth data from Hoshangabad division, no results have been eliminated in step IV and V, as all surviving results have 0.99 of R_a^2 value (Table 4) and all of their parameters are significantly different from zero (Table 7). And finally, we choose the best fit model and find that Monomolecular growth model (method VI) and four parameters Weibull growth

Table 3: Estimation of parameters along with the summary of statistical analysis to height growth data from Hoshangabad division.

Growth Models		Methods	A	*B/b/* b_1/β	*k*	*m/d*	X^2	RMSE	R^2 (in %)	R_a^2	(in %)
Monomolecular		Method I	22.1381	1.0339	.2162	---	.098	.367	99.27	.99	99.08
		Method II	18.4943	1.0724	.3052	---	.076	.390	99.17	.98	98.17
		Method III	21.7349	1.0074	.2162	---	.086	.306	99.49	.99	99.33
		Method IV	18.9265	1.0924	.3052	---	.050	.322	99.44	.99	98.93
		Method V	19.1200	1.0767	.2939	---	.047	.307	99.49	.99	99.01
		Method VI	20.2819	1.0411	.2546	---	.052	.278	99.58	.99	99.34
Gompertz		Method I	19.0618	2.5772	.4524	---	.330	.635	97.81	.96	97.33
		Method II	16.4921	2.5281	.5725	---	.191	.591	98.10	.96	96.06
		Method III	18.8731	2.4331	.4524	---	.214	.529	98.48	.97	98.03
		Method IV	17.0642	2.6513	.5725	---	.145	.508	98.59	.97	97.59
		Method V	21.8126	2.0489	.3145	---	.445	.655	97.67	.95	97.26
		Method VI	17.9474	2.3339	.4802	---	.168	.452	98.89	.98	98.38
Logistic		Method I	17.9420	7.9593	.7265	---	.675	.888	95.71	.91	94.76
		Method II	15.5795	7.2402	.8921	---	.362	.796	96.56	.93	93.10
		Method III	18.8329	7.9532	.7265	---	.708	1.099	93.43	.87	89.93
		Method IV	16.6592	8.2833	.8921	---	.345	.767	96.80	.94	95.30
		Method V	15.1116	9.3857	1.1181	---	.477	.983	94.75	.89	89.65
		Method VI	15.1956	9.2801	1.1021	---	.454	.957	95.02	.90	90.28
Weibull	4	Method I		Not Fi tted due t o singular matrix occurs during computation							
	3	Method I	19.4958	---	.23053	1.0936	.070	.309	99.48	.99	99.19
	2	Method I	21.5015	---	.2186	---	.100	.311	99.47	.98	99.32
VB	4	Method I	20.8027	14.5914	.2284	.0909	.284	.481	98.74	.97	98.41
		Method II	21.7108	10.7162	.2192	.1866	.302	.486	98.72	.97	98.56
	3	Method I	21.1250	.8467	.2137	---	.256	.457	98.87	.98	98.56
		Method II	20.3753	.8530	.2154	---	.326	.639	97.85	.97	96.44
	2	Method I	25.9146	---	.1541	---	.312	.615	97.95	.98	97.68
		Method II	26.0046	---	.1628	---	.183	.535	98.45	.98	97.61
CR	4	Method I	21.5681	.9947	.1958	.9000	.220	.424	99.02	.98	98.73
		Method II	22.4362	.9530	.1885	.9898	.274	.459	98.85	.98	98.70
	3	Method I	33.3230	---	.1133	.9132	.745	1.294	90.92	.86	82.67
		Method II	20.7444	---	.2382	1.0370	.090	.311	99.48	.99	99.25

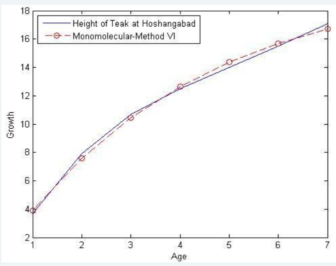

Figure 1: Observed data along with the top two results for height growth data of Hoshangabad.

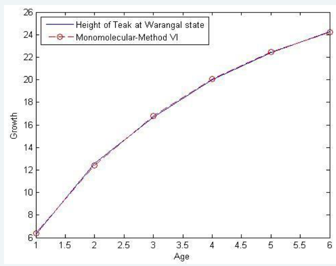

Figure 3: Observed data along with the top two results for height growth data of Warangal.

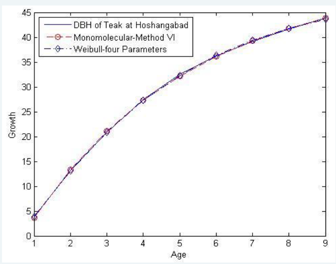

Figure 2: Observed data along with the top two results for DBH growth data of Hoshangabad.

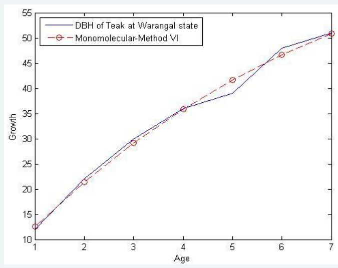

Figure 4: Observed data along with the top two results for DBH growth data of Warangal.

Citation: Mahanta DJ, Borah M. A Comparative Study of Nonlinear Growth Models on Teak (*Tectonagrandis L.*) in India. J Plant Sci Res. 2017;4(1): 165.

Table 4: Estimation of parameters along with the summary of statistical analysis to DBH growth data from Hoshangabad division.

Growth Models		Methods	A	B/b/ b_1 / β	k	m/d	X^2	RMSE	R^2 (in %)	R_a^2	(in %)
Monomolecular		Method I	51.4121	1.1665	.2308	---	.036	.297	99.95	.99	99.93
		Method II	50.6219	1.1890	.2391	---	.113	.275	99.95	.99	99.94
		Method III	51.2416	1.1728	.2308	---	.032	.202	99.98	.99	99.97
		Method IV	50.5118	1.1856	.2391	---	.075	.267	99.96	.99	99.95
		Method V	52.5322	1.1543	.2179	---	.012	.207	99.97	.99	99.97
		Method VI	51.8184	1.1635	.2246	---	.016	.185	99.98	.99	99.98
Gompertz		Method I	45.8708	2.9338	.4283	---	1.349	1.066	99.32	.99	99.26
		Method II	44.9708	3.6543	.4877	---	.683	1.001	99.40	.99	99.29
		Method III	49.8349	3.6416	.4283	---	1.703	2.006	97.59	.96	99.69
		Method IV	46.4961	3.8454	.4877	---	.930	1.377	98.86	.98	98.59
		Method V	57.6666	2.5288	.2754	---	3.393	2.301	96.83	.95	96.03
		Method VI	44.7922	3.3996	.4806	---	0.681	.869	99.55	.99	99.44
Logistic		Method I	43.9088	8.2081	.6304	---	2.512	1.634	98.40	.97	98.26
		Method II	42.5763	19.7277	.8549	---	2.952	2.166	97.19	.95	96.75
		Method III	105.7693	43.7263	.6304	---	71.8	24.18	-249.86	-4.59	-503.11
		Method IV	56.9020	30.4593	.8549	---	17.0	9.136	50.08	.20	23.18
		Method V	35.7249	37.4705	1.4975	---	5.047	4.326	88.80	.82	82.24
		Method VI	36.0556	36.8190	1.4719	---	4.702	4.178	89.56	.83	83.55
Weibull	4	Method I	48.8618	54.3209	.1887	1.1452	.009	.176	99.98	.99	99.98
	3	Method I	44.7232	---	.1134	1.5251	.307	0.632	99.76	.99	99.68
	2	Method I	66.7442	---	.1248	---	2.433	1.670	98.33	.97	98.15
VB	4	Method I	50.8787	41.7167	.2482	.0909	.061	.247	99.96	.99	99.96
		Method II	50.2319	46.4833	0.2487	0.0585	.017	.211	99.97	.99	99.96
	3	Method I	51.2114	-10.8118	.2376	---	1.004	.558	99.81	.99	99.81
		Method II	51.4894	-8.9503	.2315	---	.044	.281	99.95	.99	99.94
	2	Method I	1.5551	---	-.3671	---	787.3	32.4	-528.2	-6.18	-815.0
		Method II	-1.0440	---	-.5844	---	-690.3	5.251	14566.5	166.33	27044.5
CR	4	Method I	51.6494	1.2477	0.2246	.9000	40.29	1.209	99.12	.99	99.11
		Method II	45.5156	1.4317	.2701	.558	-6.25	2.442	102.64	1.04	102.07
	3	Method I	2042.103	---	.0062	1.1607	20.179	11.58	19.67	-.07	-41.74
		Method II	84.3317	---	.1134	1.2323	2.286	2.568	96.06	.95	94.03

model give the similar results with the $R^2_{prediction}$ and R^2 values 99.98 and 99.98 respectively. The two results are plotted in order to illustrate their differences (Figure 2). Both the results produced a very similar result for DBH growth data from Hoshangabad division.

The estimation of parameters for the growth models along with the summary of statistical analysis to height growth data from Warangal state were presented in Table 5. The eliminated results in each step were also highlighted accordingly. Here, logistic growth model (method I,V and VI) has been eliminated due to non-logical estimates of one of its parameter. The methods have estimated for the asymptote (23.8857 m, 22.9976 m and 23.1506 m respectively) smaller than the dominated height (24.30 m). Three parameters Von Bertalanffy growth model (method I and II) are also eliminated in the step I due to having negative parameter estimates. In the case of height growth data from Warangal state, it is also noticed that no results have been eliminated in step IV and V, as all surviving results have 0.99 of R_a^2 value (Table 5) and all of their parameters were significantly different from zero (Table 7). And finally, based on R^2 and $R^2_{prediction}$, the better result has been chosen and it is find as Monomolecular growth model (method VI). The observed and the estimated value are shown in Figure 3.

The estimation of parameters for the growth models and the summary of statistical analysis to DBH growth data from Warangal state are presented in Table 6. The best result is selected and found as monomolecular growth model for method VI. Figure 4 represents the observed and the estimated values. The eliminated results in each step were highlighted accordingly in the Table 6. For the DBH growth data, only eight results are promoted to step III as most of the results are failed to obtain 95% level of significance. In step IV, four results are eliminated due to having R_a^2 value less than 0.99. Plot of the observed along with the estimated result are also presented in Figure 4.

From the above discussion it is clear that the monomolecular growth model with the method VI provide the better results for all data sets. Also the four parameters Weibull growth model produces better results for DBH growth of Hoshangabad division among with the monomolecular growth model. It is also observed that, monomolecular growth model with all its methods of estimation provide a healthy fit for the data sets except the DBH growth of Warangal state. In case of DBH growth of Warangal state, the table values of χ^2 for 95% level of significance is found to be lesser than the calculated χ^2 values for two of the methods (Method II and IV). Also three of its method (Method I,III and V) is found failed to attend the 0.99 value of R_a^2.

Citation: Mahanta DJ, Borah M. A Comparative Study of Nonlinear Growth Models on Teak (*Tectonagrandis L.*) in India. J Plant Sci Res. 2017;4(1): 165.

Table 5: Estimation of parameters along with the summary of statistical analysis to height growth data from Warangal state.

Growth Models		Methods	A	$B/b/b_1/\beta$	k	m/d	X^2	RMSE	R^2 (in %)	R_a^2	(in %)
Monomolecular		Method I	29.3128	1.0605	.3007	---	.009	.139	99.95	.99	99.94
		Method II	29.7602	1.0409	.2883	---	.013	.141	99.95	.99	99.94
		Method III	29.3508	1.0559	.3007	---	.006	.115	99.96	.99	99.96
		Method IV	29.8473	1.0449	.2883	---	.010	.135	99.95	.99	99.95
		Method V	28.7071	1.0706	.3177	---	.005	.122	99.96	.99	99.93
		Method VI	29.1420	1.0608	.3062	---	.006	.113	99.97	.99	99.96
Gompertz		Method I	25.4228	2.5419	.5999	---	.067	.404	99.56	.99	99.27
		Method II	26.4567	2.3586	.5396	---	.072	.361	99.65	.99	99.58
		Method III	25.6692	2.5242	.5999	---	.050	.356	99.66	.99	99.49
		Method IV	26.8528	2.4250	.5396	---	.078	.408	99.55	.99	99.43
		Method V	31.9867	2.0577	.3597	---	.374	.836	98.14	.95	97.29
		Method VI	25.9623	2.4093	.5702	---	.056	.337	99.69	.99	99.58
Logistic		Method I	23.8857	7.1097	.9349	---	.181	.668	98.81	.97	97.90
		Method II	24.9301	6.3350	.8361	---	.170	.585	99.08	.98	98.83
		Method III	24.6159	7.2631	.9349	---	.164	.665	98.82	.97	98.30
		Method IV	26.1825	7.0499	.8361	---	.291	.892	97.87	.95	96.60
		Method V	22.9976	7.8323	1.0876	---	.217	.858	98.04	.95	95.87
		Method VI	23.1506	7.7481	1.0704	---	.200	.818	98.22	.96	96.35
Weibull	4	Method I	31.6992	36.0614	.3519	.8383	.002	.067	99.96	.99	99.95
	3	Method I	27.0873	---	.2686	1.1733	.013	.186	99.90	.99	99.85
	2	Method I	31.5325	---	.2484	---	.066	.29	99.76	.99	99.74
VB	4	Method I	29.9822	21.2525	.2857	.0909	.119	.381	99.61	.99	99.60
		Method II	30.6153	17.2375	.2807	.1516	.080	.328	99.71	.99	99.69
	3	Method I	30.2645	-.1349	.2714	---	.089	.326	99.72	.99	99.71
		Method II	30.0835	-.7938	.2754	---	.029	.221	99.87	.99	99.84
	2	Method I	34.7985	---	.1997	---	.225	.735	98.56	.98	98.18
		Method II	34.8401	---	.2124	---	.081	.451	99.46	.99	99.03
CR	4	Method I	30.6413	1.0429	.2540	.9000	.054	.253	99.83	.99	99.82
		Method II	31.1110	1.0309	.2512	.9437	.036	.216	99.88	.99	99.87
	3	Method I	106.6136	---	.0328	.7828	.710	1.652	92.74	.88	82.43
		Method II	34.0534	---	.2112	.9737	.076	.386	99.60	.99	99.44

Table 6: Estimation of parameters along with the summary of statistical analysis to DBH growth data from Warangal state.

Growth Models	Methods	A	$B/b/b_1/\beta$	k	m/d	X^2	RMSE	R^2 (in %)	R_a^2	(in %)
Monomolecular	Method I	76.000	.9849	.1567	---	.265	1.225	99.09	.98	98.65
	Method II	63.5455	.9991	.2106	---	.349	1.491	98.66	.97	97.54
	Method III	75.3216	.9779	.1567	---	.257	1.207	99.11	.98	98.66
	Method IV	64.9623	1.0144	.2106	---	.323	1.386	99.84	.98	98.12
	Method V	70.2822	.9891	.1781	---	.262	1.254	99.05	.98	98.49
	Method VI	79.0094	.9712	.1436	---	.268	1.199	99.13	.99	98.69
Gompertz	Method I	59.95030	2.3591	.3829	---	.563	1.613	98.43	.97	97.87
	Method II	51.6012	2.2763	.4702	---	.674	2.040	97.48	.95	95.08
	Method III	59.0827	2.2666	.3829	---	.455	1.502	98.64	.97	98.07
	Method IV	53.1861	2.3629	.4702	---	.529	1.790	98.06	.96	96.69
	Method V	66.2514	2.1091	.3025	---	.631	1.528	98.59	.97	98.15
	Method VI	60.7303	2.1568	.3525	---	.487	1.456	98.72	.97	98.21
Logistic	Method I	54.9474	6.7804	.6390	---	1.064	2.107	97.32	.95	96.53
	Method II	47.1306	5.9734	.7656	---	1.188	2.669	95.69	.91	91.33
	Method III	55.8389	6.5998	.6390	---	1.003	2.249	96.94	.94	95.68
	Method IV	49.6432	6.5966	.7656	---	.882	2.266	96.89	.94	94.87
	Method V	46.9508	6.7822	.8536	--	1.254	2.777	95.34	.91	91.21
	Method VI	47.2798	6.7439	0.8417	---	1.188	2.700	95.59	.91	91.81

Citation: Mahanta DJ, Borah M. A Comparative Study of Nonlinear Growth Models on Teak (*Tectonagrandis L.*) in India. J Plant Sci Res. 2017;4(1): 165.

Weibull	4	Method I	34.0000	0.0000	.0000	.9613	34.059	12.86	0.00	-1.0	-50.53
	3	Method I	73.6872	---	.1801	0.9535	.253	1.2335	99.08	.99	98.55
	2	Method I	70.8236	---	.1788	---	.291	1.274	99.01	.98	98.51
VB	4	Method I	109.1104	65.9357	.0942	.0909	.731	1.754	98.14	.96	97.07
		Method II	109.6865	57.2476	.0920	.1202	.585	1.467	98.69	.97	98.24
	3	Method I	120	5.6667	.0771	---	.694	1.735	98.18	.97	97.09
		Method II	118.817	5.6442	.0739	---	.559	1.463	98.71	.90	98.26
	2	Method I	69.3280		.1901	---	.330	1.412	98.79	.99	98.21
		Method II	69.3407	---	.1864	---	.281	1.292	98.99	.99	98.45
CR	4	Method I	140.6016	.9764	.0562	.9000	.649	1.711	98.23	.96	97.11
		Method II	140.8934	.9716	.0549	.9211	.523	1.407	98.80	.98	98.34
	3	Method I	3170.62	---	.0008	.7745	1.796	3.581	92.25	.88	84.79
		Method II	55.5152	---	.3071	1.1267	.704	2.026	97.52	.96	95.75

Table 7: 95% Confidence intervals of the parameters of surviving growth models.

Data	Model	Method	*A*		*B/b/ b_1 / β*		*k*		*m/d*	
			Lower Limit	Upper Limit	Lower Limit	Upper Limit	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Hoshangabad_height	Monomolecular	I	16.489	27.787	0.916	1.151	0.098	0.334	---	---
		III	17.014	26.456	0.912	1.103	0.113	0.319	---	---
		IV	16.284	21.569	0.952	1.233	0.191	0.420	---	---
		V	16.423	21.817	0.947	1.207	0.185	0.403	---	---
		VI	17.117	23.446	0.939	1.144	0.159	0.350	---	---
	Weibull 3	I	15.347	23.644	---	---	0.186	0.275	0.844	1.343
	Chapman Richards 3	II	16.810	24.679	---	---	0.108	0.369	0.754	1.320
Hoshangabad_DBH	Monomolecular	I	49.663	53.161	1.135	1.198	0.212	0.249	---	---
		II	49.101	52.143	1.159	1.219	0.222	0.256	---	---
		III	50.047	52.436	1.151	1.194	0.218	0.243	---	---
		IV	49.035	51.989	1.157	1.215	0.222	0.256	---	---
		V	51.181	53.883	1.133	1.175	0.205	0.231	---	---
		VI	50.673	52.964	1.144	1.183	0.213	0.236	---	---
	Gompertz	VI	42.358	47.226	2.772	4.027	0.402	0.559	---	---
	Weibull 4	I	46.640	51.084	49.996	58.646	0.159	0.219	1.023	1.268
	Von Bertalanffy 4	I	50.390	51.368	36.187	47.247	0.239	0.257	0.059	0.122
		II	49.729	50.735	39.985	53.582	0.239	0.258	0.024	0.093
Warangal_height	Monomolecular	I	27.052	31.574	1.003	1.118	0.244	0.358	---	---
		II	27.278	32.242	0.986	1.096	0.230	0.346	-	---
		III	27.473	31.229	1.009	1.103	0.253	0.348	---	---
		IV	27.475	32.220	0.992	1.098	0.233	0.343	---	---
		V	26.905	30.509	1.018	1.124	0.267	0.369	---	---
		VI	27.362	30.922	1.014	1.108	0.260	0.353	---	---
	Gompertz	VI	23.311	28.614	1.859	2.960	0.405	0.736	---	---
	Weibull 4	I	23.024	40.374	19.658	52.465	0.226	0.478	0.418	1.259
	Weibull 3	I	24.703	29.471	---	---	0.244	0.293	1.039	1.307
	Weibull 2	I	30.915	32.150	---	---	0.240	0.257	---	---
Warangal_DBH	Monomolecular	VI	38.482	119.537	0.892	1.051	0.013	0.274	---	---
	Weibull 3	I	20.880	126.494	---	---	0.065	0.295	0.590	1.317
	Von Bertalanffy 2	I	55.887	82.769	---	---	0.129	0.251	---	---
		II	56.614	82.067	---	---	0.130	0.243	---	---

Conclusion

In this paper we summarize and attempt to find the best fit growth model along with the best method of estimation for the teak growth in India based on the available teak data. A specific selection criterion with six distinct steps has been considered to compare the results. According to the results, monomolecular growth model while estimated by method VI provides the better results for all data sets whereas four parameters Weibull growth model offer similar result for DBH growth of Hoshangabad division. By observing all the results, it can be concluded that, the monomolecular growth model is more reasonable over the remaining growth models for describing the growth of Teak in India. One may consider any method of estimation (From method I to method VI) to estimate the parameters of the monomolecular growth model but the method VI is more preferable.

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